THE CHIRAL OSCILLATOR AND ITS APPLICATIONS IN QUANTUM THEORY

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Abstract
The fundamental importance of the chiral oscillator is elaborated. Its quantum invariants are computed. As an application the Zeeman effect is analysed. We also show that the chiral oscillator is the most basic example of a duality invariant model, simulating the effect of the familiar electric-magnetic duality.

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It is well known that the Harmonic Oscillator (HO) pervades our understanding of quantum mechanical as well as field theoretical models in various contexts. An interesting thrust in this direction was recently made in [1] where the quantum invariants of the HO were computed. It was also opined that this approach could be used for developing a technique [2] to study interacting and time dependent (open) systems.

In this paper we argue that, in some instances, the Chiral Oscillator (CO) instead of the usual HO captures the essential physics of the problem. This is tied to the fact that the CO simulates the left-right symmetry. Consequently the CO has a decisive role in those cases where this symmetry is significant.

The CO is first systematically derived from the HO and the issue of symmetries is clarified. Indeed, it is explicitly shown that the decomposition of the HO leads to a pair of left-right symmetric CO's. The soldering of these oscillators to reobtain the HO is an instructive exercise. Following the methods of [1, 2], the quantum invariants of the CO's are computed and their connection with the HO invariant is illuminated. As an application, the Zeeman splitting [3] for the Hydrogen atom electron energy levels under the influence of a constant magnetic field is studied. The interaction of the atom with a time-dependent magnetic field, constituting an open system, can also be analysed from the general expressions. In a completely different setting we show that the CO is the most basic example of a duality invariant theory [4]. By reexpressing the computations in a suggestive electromagnetic notation, the mapping of this duality with Maxwell's electromagnetic duality is clearly established.

The Lagrangean for the one dimensional HO is given by

$$L = \frac{M}{2}(\dot{x}^2 - \omega^2 x^2). \tag{1}$$

To obtain the CO, the basic step is to convert (1) in a first order form by introducing an auxiliary variable Λ in a symmetrised form,

$$L = \frac{M}{2}(\Lambda\dot{x} - x\dot{\Lambda} - \Lambda^2 - \omega^2 x^2). \tag{2}$$

There are now two distinct classes for relabelling these variables corresponding to proper and improper rotations generated by the matrices with determinant ± 1,

$$\begin{pmatrix} x \\ \frac{\Lambda}{\omega} \end{pmatrix} = \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \begin{pmatrix} x \\ \frac{\Lambda}{\omega} \end{pmatrix} = \begin{pmatrix} sin\phi & cos\phi \\ cos\phi & -sin\phi \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

leading to the structures,

$$L_\pm = \frac{M}{2}(\pm\omega\epsilon_{\alpha\beta}x_\alpha\dot{x}_\beta - \omega^2 x_\alpha^2), \tag{3}$$

where $\alpha = 1, 2$ is an internal index with $\epsilon_{12} = 1$. The basic Poisson brackets of the above model are read off from the symplectic structure,

$$\{x_\alpha, x_\beta\}_\pm = \mp\frac{1}{\omega M}\epsilon_{\alpha\beta}. \tag{4}$$

The corresponding Hamiltonians are,

$$H_\pm = \frac{M\omega^2}{2}(x_1^2 + x_2^2) = \tilde{H}. \tag{5}$$

The above Lagrangeans in (3) are interpreted as two bi-dimensional CO's rotating in either a clockwise or an anti-clockwise sense. A simple way to verify this property is to look at the spectrum of the angular momentum operator,

$$\omega J_{\pm} = \omega \epsilon_{\alpha\beta} x_\alpha p_\beta = \pm \frac{1}{2} M \omega^2 x_\alpha^2 = \pm \tilde{H}, \tag{6}$$

where $\tilde{H}$ is defined above.

To complete the picture it is desirable to show the mechanism of combining the left and right CO's to reproduce the usual HO. This is achieved by the soldering technique [5, 6] introduced recently. Let us then begin with two *independent* chiral Lagrangeans $L_+(x)$ and $L_-(y)$. Consider the following gauge transforms, $\delta x_\alpha = \delta y_\alpha = \eta_\alpha$ under which

$$\delta L_{\pm}(z) = M\omega \epsilon_{\alpha\beta} \eta_\alpha (\pm \dot{z}_\alpha + \omega \epsilon_{\alpha\beta} z_\beta), \quad z = x, y.$$

Introduce a new variable B_α, which will effect the soldering, transforming as, $\delta B_\alpha = \epsilon_{\alpha\beta}\eta_\beta$. This new Lagrangean

$$L = L_+(x) + L_-(y) - M\omega B_\alpha (\dot{x}_\alpha + \omega \epsilon_{\alpha\beta} x_\beta - \dot{y}_\alpha + \omega \epsilon_{\alpha\beta} y_\beta), \tag{7}$$

is invariant under the above transformations. Eliminating B_α by the equations of motion, we obtain the final soldered Lagrangean,

$$L(w) = \frac{M}{4}(\dot{w}_\alpha^2 - \omega^2 w_\alpha^2),$$

which is no longer a function of x and y independently, but only on their gauge invariant combination, $w_\alpha = x_\alpha - y_\alpha$. The soldered Lagrangean just corresponds to a bi-dimensional simple harmonic oscillator. Thus, by starting from two distinct Lagrangeans containing the opposite aspects of chiral symmetry, it is feasible to combine them into a single Lagrangean.

The connection between the CO and HO is now used to obtain the invariants of the former by exploiting known results [1] for the latter. For the positive CO,

$$I^+ = \frac{1}{2}\tan^{-1}({x_1}^{-1} x_2) + \frac{1}{2}\tan^{-1}(x_2 {x_1}^{-1}), \tag{8}$$

is the invariant, while I^- is given by interchanging x_1 and x_2. Note that non-commutativity of the variables has already been taken into account. Incorporating the "soldering" prescription [6] whereby we were able to construct a bi-dimensional oscillator from the two CO's, another quantum invariant can also be obtained,

$$I^+(x_1, x_2) \oplus I^-(y_1, y_2) = I(x_1 - y_1, x_2 - y_2), \tag{9}$$

where, the right hand side of the equation is a simple sum of two terms, obtained by substituting $x_1 - y_1$, $M(\dot{x}_1 - \dot{y}_1)/2$ and $x_2 - y_2$, $M(\dot{x}_2 - \dot{y}_2)/2$ in place of x and p in the corresponding expression for HO in [1]. We stress that the above invariant operators are independent as they pertain to completely different systems and were not present in the literature so far. In the next section, we will put the CO invariants into direct use in interacting and open quantum systems by considering the Zeeman effect.

Let us consider the simplistic Bohr model of Hydrogen atom, where the (non-relativistic) electron is moving in the presence of a repulsive centrifugal barrier and the attractive Coulomb potential. The effective central potential has a well like structure and we consider the standard HO approximation about the potential minimum. The excitations are the HO states above the minimum. Hence the electron, at a particular stationary state, is approximated to an oscillator with a frequency ω, obtained from the effective potential seen by the atomic electron without the magnetic field.This yields $\omega = (Me^4)/l^3$, with $l = Mr^2\dot{\phi}$ being the angular momentum, when expressed in plane polar coordinates.

In the presence of a magnetic field $\mathbf{B}$, the motion of the electron can be broken into components parallel and perpendicular to $\mathbf{B}$. The Lorentz force acting on the electron affects the motion in the normal plane of $\mathbf{B}$ only, the motion being two rotational modes in the clockwise and anti-clockwise sense, or more succintly two CO's of opposite chirality. In this setup, $\mathbf{B}$ splits the original level into three levels, one of frequency ω remaining unchanged and the other two frequencies changed to $\omega \pm (eB)/(2Mc)$ [3]. This clearly shows that there is a redundancy in the number of degrees of freedom in treating the electron as a HO, whereas the CO representation is more elegant and economical whenever the degeneracy between the right and left movers is lifted such as in the presence of magnetic field.

The Hamiltonian of a charged HO in an axially symmetric magnetic field is,

$$\mathbf{A} = \frac{1}{2}B(t)\mathbf{k}\times\mathbf{r}, \quad \mathbf{B}(t) = \nabla\times\mathbf{A} = B(t)\mathbf{k}$$

$$H = \frac{1}{2M}(\mathbf{p} - e\mathbf{A})^2 + \frac{1}{2}M\omega^2 r^2 = \frac{1}{2M}({p_1}^2 + {p_2}^2) + \frac{1}{2}M\omega^2({x_1}^2 + {x_2}^2)$$
$$+ \frac{eB(t)}{2Mc}(x_2p_1 - x_1p_2) + \frac{e^2}{8Mc^2}B(t)^2({x_1}^2 + {x_2}^2). \tag{10}$$

For the semi-classical reasoning (regarding the Zeeman effect) to hold, $|\ \mathbf{B}\ |$ must be small in the sense that the radius of gyration $r = (cMv)/(eB) = (cl)/(eBr)$, which simplifies to $r = \sqrt{(cl)/(eB)} = \sqrt{(nc\hbar)/(eB)}$ is much larger than the Bohr radius of the (Hydrogen) atom [7] $r_{Bohr} = \hbar^2/(Me^2)$. This condition is expressed as

$$\frac{\hbar^3 B}{cM^2e^3} << 1. \tag{11}$$

In our Hamiltonian (10), this condition will hold if

$$|\ \frac{1}{2}M\omega^2({x_1}^2 + {x_2}^2)\ | >> |\ \frac{eB(t)}{2Mc}(x_2p_1 - x_1p_2)\ |\ . \tag{12}$$

To verify this, substitute $\omega = Me^4/l^3$ and $({x_1}^2 + {x_2}^2) = r_{Bohr}$ in the left hand side, and $(x_2p_1 - x_1p_2) = l$ in the right hand side. This reproduces (11). The quadratic B-term in (10) is still smaller.

The above structure of the Hamiltonian is very similar to the model of a charged particle in a specified electromagnetic field, considered in [1, 2]. The idea there is to look for the invariants of the full interacting Hamiltonian, and to construct eigenstates of the

invariant operator. The solutions of the time dependent Schrodinger equation are related uniquely to these eigenstates via a time dependent phase,

$$\mid \lambda, k, t >_{Sch} = e^{i\alpha_{\lambda k}(t)} \mid \lambda, k, t >_I, \quad I(t) \mid \lambda, k, t >_I = \lambda \mid \lambda, k, t >_I,$$

satisfying,

$$i\hbar \frac{d\alpha_{\lambda k}}{dt} = < \lambda, k \mid_I (i\hbar \frac{\partial}{\partial t} - H) \mid \lambda, k >_I .$$

Next we define,

$$H_o = \frac{1}{2M}({p_1}^2 + {p_2}^2) + \frac{1}{2}M\omega^2({x_1}^2 + {x_2}^2)$$

and the rest of the B-dependent terms appearing in (10) as small perturbations. In the framework of [8], the invariant operator is also expressible as a power series in the small parameter $B\hbar^3/(cM^2e^3)$ and the zeroth order invariant I_0 is identical to H_o. Hence the eigenstates of H_o and I_o will be same and $\mid \lambda, k, t >_I = exp(-i(n + \frac{1}{2})\omega t) \mid \lambda, k >_I$. As in the conventional scenario, the total energy is also expressed as a series with the zeroth term being $(n + \frac{1}{2})\hbar\omega$. Thus we will compute the B-dependent corrections only by the scheme of [2], which actually comprises the task of calculating the phase $\alpha_{\lambda k}$. Here the CO's will come into play.

As we have already established the connection between the results of HO and CO models, we simply replace the HO variables by the CO ones in the final result. From the symplectric structure, the following identifications are consistent,

$$CO^+ : \quad \{{x_1}^+, \ {x_2}^+\} = -\frac{1}{\omega M}, \quad \rightarrow p_1 \equiv -\omega M {x_2}^+, \ p_2 \equiv \omega M {x_1}^+, \tag{13}$$

$$CO^- : \quad \{{x_1}^-, \ {x_2}^-\} = \frac{1}{\omega M}, \quad \rightarrow p_1 \equiv \omega M {x_2}^-, \ p_2 \equiv -\omega M {x_1}^-. \tag{14}$$

Introducing these in (10), we get,

$$H_\pm = \frac{M}{2}({x_1}^2 + {x_2}^2)(1 + \frac{e^2B^2}{4M^2c^2} \mp \frac{eB}{Mc}). \tag{15}$$

The above splitting in the energy is one of our main results. This underlines the economy in the CO formulation since one CO is sufficient to obtain the correct results. Obviously it is easier to work with less number of degrees of freedom in cases of more complicated systems. Essentially this change in the relative sign of the linear B term can also be interpreted as a consequence of the opposite angular momenta of the CO's, as demonstrated before. This brings us to the cherished expression of the phase for the two CO's,

$$\alpha^\pm_{jn} = \mp[n + (j + \frac{1}{2})]\frac{e}{Mc}\int^t dt'[\frac{1}{2}B(t') - \rho^{-2}(t')], \tag{16}$$

where the quantum numbers j and n are explained in [2] and $\rho(t')$ satisfies the equation,

$$(\frac{Mc}{e})^2\ddot{\rho} + \frac{B^2(t)}{2}\rho - \rho^{-3} = 0.$$

Considering the simplest case, that is normal Zeeman effect, where B is a constant, we find a time-independent solution of ρ, $\rho^2 = \pm\sqrt{2}/B$. When $\rho^2 = -\sqrt{2}/B$ is substituted in (16), the standard Zeeman level splitting is reproduced.

$$E_n^{\pm} = (n + \frac{1}{2})\hbar\omega \pm [n + (j + \frac{1}{2})]\frac{eB}{Mc}. \tag{17}$$

On the other hand, $\rho^2 = \sqrt{2}/B$ reveals no shift in the energy eigenvalue. Clearly this is reminiscent of the fact that the energy of the mode parallel to **B** remains unaffected. For time dependent magnetic field, one has to obtain the appropriate solution for ρ. Inserting this in (16) it is then possible to obtain the solutions of the corresponding Schrödinger equation.

We next show the possibility of interpreting the CO as a prototyype of a duality invariant model characteristic of the electric-magnetic duality [4]. For convenience, we set $M = \omega = 1$ in (1). Introduce a change of variables, $E = \dot{x}, \quad B = x$, so that

$$\dot{B} - E = 0 \tag{18}$$

is identically satisfied. In these variables, the Lagrangian (1) and the corresponding equation of motion are expressed as

$$L = \frac{1}{2}(E^2 - B^2), \quad \dot{E} + B = 0. \tag{19}$$

It is simple to observe that the transformations, [2] $E \to \pm B;\ B \to \pm E$, swap the equation of motion in (19) with the identity (18) although the Lagrangean (19) is not invariant. The similarity with the corresponding analysis in Maxwell theory is quite striking, with x and $\dot{x}$ simulating the roles of the magnetic and electric fields, respectively. There is a duality among the equation of motion and the "Bianchi" identity (18), which is not manifested in the Lagrangean.

Let us now consider the Lagrangean for the CO,

$$L_{\pm} = \frac{1}{2}(\pm\epsilon_{\alpha\beta}x_\alpha\dot{x}_\beta - x_\alpha^2) = \frac{1}{2}(\pm\epsilon_{\alpha\beta}B_\alpha E_\beta - B_\alpha^2). \tag{20}$$

These chiral Lagrangeans are manifestly invariant under the duality transformations,

$$x_\alpha \to R^+_{\alpha\beta}(\theta)x_\beta. \tag{21}$$

Thus, the CO's represent a quantum mechanical example of a duality invariant model. Indeed, the expressions for $L_\pm$ given in the second line of (20) closely resemble the analogous structure for the Maxwell theory deduced in [9].

The generator of the infinitesimal symmetry transformmation is given by, $Q = x_\alpha x_\alpha/2$, so that the complete transformations (21) are generated by,

$$x_\alpha \to x'_\alpha = e^{-i\theta Q}x_\alpha e^{i\theta Q} = R^+_{\alpha\beta}(\theta)x_\beta.$$

This follows by exploiting the basic bracket of the theory given in (4).

[2] Note that these are the discrete cases ($\theta = \pm\frac{\pi}{2}$) for a general $SO(2)$ rotation matrix parametrised by the angle θ.

To conclude, certain interesting properties of the CO were illustrated. A systematic method of obtaining this oscillator from the usual simple HO was given. It was also shown that the distinct left and right components of the CO were combined by the soldering formalism [5, 6] to yield a bi-dimensional HO. In this way the symmetries of the model were highlighted. The importance in the CO lies in the fact that in some cases it has a concrete and decisive role than the usual simple HO in illuminating the basic physics of the problem. This was particularly well seen in the derivation of the Zeeman splitting by exploiting the perturbation theory technique based on quantum invariant operators [1, 2]. An explicit computation of the quantum invariants for the CO was also performed. Apart from the study of the Zeeman effect, such CO invariants can find applications in other quantum mechanical examples, particularly where a left-right symmetry is significant. Another remarkable feature of the present analysis has been the elucidation of the fundamental nature of the duality symmetry currently in vogue either in quantum field theory or the string theory [4, 9]. It was shown that the CO was a duality symmetric model, contrary to the usual HO. Expressed in the "electromagnetic" notation, this difference was seen to be the origin of the presence or absence of duality symmetry in electrodynamics.

It may be remarked that the explicit demonstration of duality symmetry in a quantum mechanical world is nontrivial since conventional analysis [4] considers two types of duality invariance confined to either $D = 4k$ or $D = 4k+2$ dimensions, thereby leaving no room for $D = 1$ dimension. Nevertheless, since most field theoretical problems can be understood on the basis of the HO, it is reassuring to note that the origin of electromagnetic duality invariance is also contained in a variant of the HO- the chiral oscillator. Our study clearly reveals that the CO complements the usual HO in either understanding or solving various problems in quantum theory.

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